WHY "CANTORIAN" ARGUMENTS AGAINST THE EXISTENCE OF GOD DON'T WORK

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Abstract. Recent attacks on God's omniscience employ a metaphysical application of Cantor's theorem. Two variations of this atheistic "Cantorian" argument can be distinguished. A quantificational form of the argument can be demonstrated to be invalid employing a defensive strategy championed by Plantinga. Turning the tables on an argument used to dismiss the failure of Cantor's theorem within mathematical systems such as Quine's New Foundations, it can be shown that a set-theoretical form of the argument is question-begging. Such atheistic "Cantorian" arguments are not only philosophically untenable, but also historically uninformed since the resources for answering them are contained within Cantor's own writings about the infinite and its relation to theology.

Theological reflection guided Georg Cantor in his mathematical research into the nature of the Transfinite. In a letter written in 1888 to the neo-Thomist priest Ignatius Jeiler, Cantor warned:

> In any case it is necessary to submit the question of the truth of the Transfinitum to a serious examination, for were it the case that I am right in asserting the truth or possibility of the Transfinitum, then (without doubt) there would be a sure danger of religious error in holding the opposite opinion, for: error circa creaturas redundat in falsam de Deo scientiam ("A mistake regarding creatures leads to a spurious knowledge of God") (Summa Contra Gentiles II,3).

Cantor's religious convictions, moreover, sustained his confidence in his research in the face of a hostile reception to it by eminent mathematicians. Henri Poincare disparaged Cantorianism as a "disease" from which mathematics would have to recover, and Cantor's arch-rival Leopold Kronecker regarded Cantor as a charlatan and a "corrupter of youth". David Hilbert, however, predicted that "from the paradise created for us by Cantor, no one will drive us out". Hilbert's opinion prevailed. Today Cantor's ideas on the infinite are almost universally regarded among mathematicians as among the most brilliant and beautiful ideas in the history of mathematics.

It is therefore ironic that "Cantorian" arguments about the nature of the Transfinite have recently been appropriated by some contemporary philosophers of religion in an attempt to discredit the notion of omniscience and so to disprove the existence of God. In this paper I show why such atheistic "Cantorian" arguments fail.

Section 1. TWO FORMS OF THE ATHEISTIC "CANTORIAN" ARGUMENT.

The essentials of the Cantorian argument occur in Bertrand Russell's Principle of Mathematics. Russell, the foremost atheist of the twentieth century, however, did not press the obvious Cantorian

arguments against divine omniscience. Perhaps this was because Russell, struggling to formulate his own emerging theory of types, realized that the Cantorian difficulties were essentially logical rather than theological.

The most persistent contemporary philosopher to insist on atheistic Cantorian arguments is Patrick Grim, who claims to give "the cleanest and most concise form" of the argument against omniscience:

> By definition, an omniscient being would have to know all truths. But there can be no set of all truths. Thus at the very least there could be no set of all that an omniscient being would have to know. If for any being there is a corresponding set of things it knows, there can be no omniscient being.

Filling in the implicit premises, we can set forth the atheistic Cantorian argument as follows:

1. If God exists, then God is omniscient.
2. If God is omniscient, then, by definition, God knows [the set of] all truths.
3. If Cantor's theorem is true, then there is no set of all truths.
4. But Cantor's theorem is true.
5. Therefore, God does not exist.

Here a brief discussion of the premises and conclusion of this argument is in order. Premise (1) could be strengthened if omniscience is assumed to be an essential divine attribute; however, this weaker premise will suffice. Premises (2) has two renditions. When God's omniscience, by definition, implies that God knows all truths, we obtain what we shall call the 'quantificational' form of the argument. If, on the other hand, God's omniscience, by definition, implies that God knows the set of all truths, we obtain what we shall call the 'set-theoretical' form of the argument. These two variations of the argument are discussed in sections 2 and 3 below. Premise (3) contains the Cantorian core of the argument. The Cantorian diagonal argument for this premise will be set forth and critically discussed below. The term 'Cantor's theorem' in premise (4) is used ambiguously: is it supposed to refer to a mathematical theorem or is it to be regarded as some immutable metaphysical truth? Although he somewhat misleadingly characterizes his argument as a "solid result", Grim understands 'Cantor's theorem' not to be referring to a mathematical theorem but to his own metaphysical application of this result. This philosophical application, claims Grim, is not "merely metaphorical" but "a powerful piece of reasoning that can claim to be fully philosophical in its own right." Finally, in fairness to Grim, we should note that he officially states his conclusion conditionally: "if for any being there is a corresponding set of things it knows, there can be no omniscient being." Grim himself, however, states that his argument is only "initially a more conditional one." Grim is therefore confident that his conditional disproof of omniscience can be strengthened to an unconditional one.

Let's critically examine the Cantorian core of the atheistic argument contained in premise

(3).
Cantor laid the foundation for the mathematical theory of the infinite by demonstrating that infinite sets, while essentially different from finite sets, nevertheless share with finite sets the property of being determinable by well-defined cardinal numbers. Cantor showed how these cardinal numbers, in turn, could be mathematically characterized using the mathematical notion of 1-1 correspondences applicable to finite and infinite sets alike.

In intuitive set theory, Cantor's theorem states:

> There is no one-to-one (1-1) correspondence between any set and its power set or set of all its subsets.

Expressed in terms of cardinality, Cantor's theorem asserts:

> Every set is cardinally smaller than its power set .

Cantor's theorem implies that the iteration of the power set operation for infinite sets leads to an ever-increasing Transfinite hierarchy of infinities.

Here it is useful to cite one more mathematical fact, the Schröder-Bernstein theorem. This theorem, which was communicated by Dedekind to Cantor in a letter dated 1899, states that if there is a 1-1 function from A into B and a 1-1 function from B into A, then A is cardinally similar to B. Intuitively, there will be a 1-1 function from any set S into a subset of its power set, namely the subset consisting of all the single element subsets of S. Assuming the existence of this intuitive mapping and the Schröder-Bernstein theorem, Cantor's theorem implies there can be no 1-1 function from P(S) into S.

So much then in rehearsal of the mathematics of Cantor's theorem. What about its alleged metaphysical application? Grim states his argument against a set of all truths succinctly:

> Suppose there is a set of all truths T..., and consider further all subsets of T, elements of the power set P(T).... To each element of this power set will correspond a truth. To each set of the power set, for example, [some specific truth] t1 either will or will not belong as a member. In either case we have a truth.... There will then be at least as many truths as there are elements of the power set P(T). But by Cantor's power set theorem, the power set of any set will be larger than the original. There will then be more truths than there are members of T; some truths will be left out.

Notice again that the term 'Cantor's power set theorem' is used here without any clear and definite meaning. If the term is supposed to refer to a mathematical result, then the premise is ambiguous and incomplete. Cantor's theorem, for example, holds in Zermelo-Fraenkel set theory but fails in Quine's New Foundations. (This is explained in more detail below.) If, on the other hand, 'Cantor's power set theorem' is to be regarded as a metaphysical truth, then it isn't sufficient merely to rehearse the Cantorian argument using the metaphysical notion of truth in place of a mathematical notion of set.

Any philosophical justification of Cantor's theorem would have to go beyond the mere fact

of its theoremhood in some consistent axiomatization of Cantorian set theory. One might try to justify Cantor’s theorem by arguing that the axioms of Cantorian set theory as formulated in, say, Zermelo-Fraenkel set theory (ZF) are metaphysically true. However, as we shall note below, even the standard way of understanding the standard model of ZF presupposes the possibility of quantifying over totalities which can not exist within the theory and to which Cantor’s theorem does not apply.

Section 2. THE QUANTIFICATIONAL FORM OF THE ARGUMENT.

Like many contemporary atheistic arguments, the argument we are examining implicitly insists that the question of the coherence of theism be raised prior to the question of the evidence for theism. The implication is that if the answer to the question of coherence is negative, then the second question becomes moot. No amount of evidence can establish that an incoherent state of affairs obtains. To adopt the stringent principle that we could rationally believe a proposition only if we are able to demonstrate that it is not incoherent would itself be irrational.

It turns out, however, that the atheistic Cantorian argument suffers from more severe philosophical problems. Philosophical and meta-mathematical mistakes cripple the core of the metaphysical application of the Cantor’s theorem, and these mistakes either invalidate the argument or disable it by requiring question-begging premises.

One way of demonstrating that the quantificational form is invalid would be to borrow a defensive strategy championed by Plantinga in his celebrated work on the free will defense. Given the propositions

G: God exists and is omniscient

and

C: Cantor’s theorem is true

we will exhibit a proposition Z that is logically compatible with both the general theistic position and consistent with the propositions G and C:

à (G Ù C Ù Z) .

Furthermore, it will turn out that G and C and Z jointly imply the proposition K that God knows all truths:

(G Ù C Ù Z) -3 K .

It will then follow by a theorem of modal logic that God’s knowing all truths is logically compatible with Cantor’s theorem:

à(C Ù K) .

Such a defense would establish the invalidity of the quantificational form of the Cantorian argument against omniscience.

Now one candidate for the proposition Z would be the following:

Z: Both truths and sets can be viewed as having an internally articulated structure iteratively built up from less complex truths or sets by various operations and unions at limits.

What proposition Z envisages is that we begin with any initial collection of basic truths, and then iteratively build up further truths in stages applying standard logical operations at successor ordinals and taking unions at limit ordinals. Cantorian set theory as characterized by ZF is in fact based on the corresponding iterative conception of set.

As Plantinga has taken pains to point out with regard to this strategy, the plausibility or implausibility of Z does not undermine its utility for the purposes of a logical defense. It turns out, however, that the view that abstract entities have an iterative structure such as that articulated in ZF is not only pondered but philosophically defended in Christopher Menzel's "theistic activist" account of the abstract universe.

Given the metaphysical possibility of such an iterative theory of truth, we may assert the compossibility of God's existence and omniscience, Cantor's theorem, and the iterative conception of truth:

à(G Ù C Ù Z) .

Since Cantor's theorem holds in an iterative theory of truth with a ZF-like structure, there would be no set of all truths within the theory. Yet such a conception of truth would still allow for quantification over all truths. The non-existence of a set of all sets does not entail the impossibility or incoherence of quantification over the universe of all sets in ZF. Similarly, the non-existence of a set of all truths does not entail the impossibility of God's knowing all truths.

Let us assume, for sake of argument, that omniscience is to be defined as knowledge of all truths. Then God's being omniscient implies that God knows all truths:

(G Ù C Ù Z) -3 K .

It follows by a theorem of modal logic from the above two premises that the truth of Cantor's theorem for truths is logically compatible with God's knowing all truths:

à(C & K) .

The quantificational form of the atheistic Cantorian argument is therefore invalid.

It might be objected that the foregoing logical defense is modally ambiguous. The modal notions, so the objection goes, are used either epistemologically or metaphysically. If they are used epistemologically ("for all we know, it is possible that..."), then the premises of the argument are too weak since the epistemological compossibility of G and C and Z is not sufficient to logically imply

K. If the modal notions are used metaphysically ("it is really possible that..."), then, so the objection goes, the premises of the argument are too strong. Assuming a background modal logic as strong as (S5), for example, it is natural to assume that metaphysical truths about the structure of truths--like mathematical truths about the structure of sets--are necessarily true if possible at all. The logical defense would then be open to a charge of question-begging: it seems to assume as a premise something that implies the controversial claim that Z is in fact true.

This objection (and similar objections which I believe can be raised against Plantinga's original free will defense) derive from the assumption of a strong modal logic in which the concession of a possibility premise may have unintentionally strong metaphysical consequences. We can, nevertheless, still escape between the horns of the objector's dilemma since the defense can be cast in terms of logical possibility alone. The essentials of the quantificational argument are as follows:

1. By definition, any omniscient being would have to know all truths.
2. But there can be no set of all truths.
3. Thus, there can be no omniscient being (because the notion of omniscience is incoherent).

But now the above illustration from set theory can be regarded as a direct counterexample to the validity of the quantificational form of the Cantorian argument:

1. By definition, quantifiers in ZF set theory would have to range over all sets.
2. But there can be no set of all sets in ZF.
3. Therefore, quantification in ZF is incoherent (because the notion of a set of all sets is incoherent).

Here the premises are true but the conclusion is false. In the standard model of ZF quantifiers range over a universe of all sets, even though within ZF--precisely because of Cantor's theorem--there is no set of all sets. Given the minimal assumption that the standard way of understanding the standard model of ZF is at least coherent, it follows that a set of all sets is apparently not needed to make sense of the notion of quantification over all sets. Similarly, we may argue that neither is a set of all truths needed to make sense of the notion of omniscience as knowledge of all truths.

Consequently, the Cantorian argument against omniscience in its quantificational form fails even to be valid. The above considerations, in fact, yield a more positive result: the consistency of a plausible theistic position can be established relative to a widely accepted understanding of the standard model of Cantorian set theory.

Section 3. THE SET-THEORETICAL FORM OF THE ARGUMENT.

Turning our attention to the set-theoretical form of the atheistic Cantorian argument, we can give an obvious rejoinder. Why must omniscience be defined in terms of a set of all truths? Suppose it turns out to be analytically true that there is no set of all truths. Then the fact that God could not form such a set would be no more of an epistemic shortcoming of God than it would be for God not being able to form within ZF the (non-existent) Russellian set supposedly containing all and only those sets that don't contain themselves or for God not to be acquainted with the (non-existent) King of France in 1905 A. D. Surely it is merely an artifact of the way some philosophers have explicated the notion of omniscience that an omniscient God would have to know a set of truths, whether that set included all truths or not. In its set-theoretical form, the Cantorian argument can quite naturally be construed as a reductio ad absurdum of the philosopher's attempt to explicate the notion of omniscience in terms of knowing a Cantorian set of truths.

There are, moreover, telling logical and meta-mathematical reasons for doubting the cogency of the atheistic Cantorian argument. The strategy of using Cantor's theorem to disprove a universal set of all truths is patently question-begging: it turns out that the premises used in the metaphysical Cantorian core actually imply that Cantor's theorem is inapplicable to that which God knows. Even on the assumption that that which God knows forms a set, the premises used in the Cantorian core imply that Cantor's theorem could not apply to such a set. Such a set would have to be non-Cantorian, i.e. a set that exists in some set theory but doesn't exist according to ZF.

If we assume that all sets are Cantorian and assume certain auxiliary philosophical assumptions about propositions, truths, and concepts, we can show that there is no set of all sets (no set of all propositions, no set of all truths, no set of all concepts, etc.). On the other hand, if we assume that such sets do exist, then we can show that they must be non-Cantorian. The situation is analogous to the child's conundrum about what happens when an irresistible force (here Cantor's diagonal construction) meets an immovable object (for example, the universal set of all truths). To embrace the latter alternative is to eliminate--or at least seriously to restrict the range of application--of the former. In its set-theoretical form the atheistic Cantorian argument can therefore be dismissed as a misdirected and question-begging reductio: it is equally plausible (if not more plausible) to construe the argument as a reductio against either the set-theoretical explication of divine omniscience or the indiscriminate application of Cantor's theorem to what God knows.

How, the objector might ask, could Cantor's theorem fail? One way to see how Cantor's theorem could fail is to examine how it does in fact fail in Quine's New Foundations [NF]. Motivated

by Russell's Simple Theory of Types, Quine's essential idea for NF concerns a simple modification of the set abstraction axiom. In NF every stratified condition j(x) yields a set. A condition j is said to be stratified if there is some assignment of numbers to the terms of ø such that, for every occurrence of 'Î' in j, the number of the term immediately following is the successor of (one more than) the number of the term immediately preceding 'Î'. The numbers of terms flanking the identity sign '=' or occurring in ordered pairs, on the other hand, must be the equal.

The universal set V exists in NF since the condition "y = y" is clearly stratified. Every set is a member of the universal set V, so it follows that VÎV. The power set condition is also stratified, and so the power set of any set exists. An immediate consequence of these two facts is that the power set of the universal set P(V) is identical to the universal set, i.e. P(V) = V. The universal set must be non-Cantorian, and Cantor's must theorem fail.

In NF it turns out that the universal set V is cardinally similar to its power set P(V) but is not cardinally similar to the set of all its one element subsets Pu(V). The "obvious" 1-1 correspondence of intuitive set theory between a set and its single element subsets does not exist in NF. By a variant of the standard Cantorian argument applied to Pu(V) and P(V), however, it follows that the cardinality of Pu(V) is less than that of V (i.e. we have that Pu(V) < P(V) = V).

A proof of Cantor's theorem by means of the Schröder-Bernstein theorem requires two conditions:

(A) every set is cardinally similar to the set of all its one-element subsets ,

and

(B) the set of all one-elements subsets of a set is cardinally less than the set of all its subsets .

In NF Cantor's theorem fails because (B) holds, but (A) does not.

Those who find NF artificial and strongly counter-intuitive might be tempted to reply as follows:

> Clearly (A) is intuitively acceptable if and only if (B) is intuitively acceptable. Since NF advocates hold (B), they are "intuitively" committed to the inconsistent Cantorian consequence that V is in fact cardinally smaller than P(V). NF does not avoid the Cantorian argument but merely cripples our ability to express it.

But then a precisely parallel argument can be directed against the metaphysical core of the atheistic "Cantorian" argument:

> Consider the following two intuitive truths about the set of all truths T required for Cantor's theorem:
>
> (A) For every distinct truth of T, there will be a distinct subset of T consisting of that truth alone

and

(B) For every distinct subset of T, there will be a distinct truth, for example, the truth that that subset is a subset of truths .

Now it seems clear that (A) is intuitively acceptable if and only if (B) is intuitively acceptable. (Grim, in fact, asserts both.) However, if both (A) and (B) were true, it would imply that Cantor's theorem fails to be applicable to a set of all truths. By the Schröder-Bernstein theorem, the conjunction of (A) and (B) implies that T is cardinally similar to P(T).

A set of all truths, if it exists, must be non-Cantorian.

One may have suspected all along that Grim atheistic "Cantorian" argument plays fast and loose with our metaphysical intuitions about truth. Our analysis confirms this suspicion. In the absence of any definite philosophical proposal about the structure of truths and truths about truths, the premises used in the "Cantorian" core can just as easily be marshalled to show that Cantor's diagonal construction cannot apply to a universe of all truths as they can to show that no such universe exists.

Section 4. SELF-REFLEXIVE INCONSISTENCIES.

Let us grant for sake of argument that Grim is correct in assuming that God's knowing all truths implies the existence of a set of all truths. By parity of argument, it would then follow that the legitimacy of propositional quantification would imply the existence of the set of all propositions. But then as Grim himself admits:

> One at least apparent difficulty is this. All of the outlines above rely on universal propositional quantification. But the only formal semantics for quantification we have is in terms of sets, and thus the only formal semantics we have for propositional quantification is in terms of a set of all propositions.

The difficulty alluded to here is that Grim himself wants to claim that his "Cantorian" arguments show there is no set of propositions, no set of all truths, no set of all concepts. If Grim arguments were successful, however, their success would seem to undermine his ability to state the very conclusions his arguments were supposed to establish.

Consider the following propositions:

(5) There is no set of all truths.

(6) Any proposition (such as the proposition that God is omniscient) which entails that there is a set of all truths is false.

(7) Nothing instantiates an incoherent concept.

The view that Cantorian arguments actually disprove a totality of truths, propositions, sets, concepts, etc. would seem to render the above propositions unassertible. For to assert (5), (6), and (7) is to assert:

(5’) for all x, if x is a set, then x is not a set of all truths ;

(6’) for all x, if x is a proposition and x entails that there is a set of all truths, then x is false ;

(7’) for any x and for any concept j, if j is incoherent, it is not the case that x instantiates j .

These statements involve universal quantification over sets, truths, propositions, objects, and concepts.
Furthermore, to assert that God does not exist, i.e., for all x, if x exists, then x is not God, requires universal quantification over the totality of all things--including propositions, sets, truths, objects, and concepts.

Grim has therefore painted himself into a Cantorian corner. If his “Cantorian” arguments were to succeed, they would show that there is, for example, no universal propositional quantification and so Grim philosophical conclusions could not even be coherently stated. Grim rejection of propositional quantification on the basis of “Cantorian” arguments, rather than being a reductio ad absurdum of the coherence of divine omniscience, would seem then to be a reductio of the coherence of Grim unbridled Cantorian intuitions.

Grim attempts to address this objection in a brief section before the concluding remarks of Chapter 4 of his book. There he claims that the self-reflexive inconsistency of his position is “merely apparent because [his] central denials--the denial that there can be any set of all truths, for example, or that there can be a totality of propositions--are emphatically not to be understood in the quantificational terms above.” Instead Grim asserts:

> [T]he denial that there is any such thing as “all truths” or “all propositions” should not itself be thought to commit us to quantifying over all truths or all propositions, any more than the denial that there is such a thing as “the square circle” should be thought to commit us to referring to something as both square and a circle.

It should be clear, however, that this reply is not only ad hoc but also misses the point.

Grim seems to be assuming that rendering occurrences of phrases like ‘the set of all truths’ in scare-quotes would block quantification over the supposedly illegitimate totalities. But the problem is not that the term ‘the set of all truths’ has no reference, rather it is that the quantification involved in expressing Grim central philosophical claims, has, according to his own position, no coherent semantics.
If standard Russellian theories of definite descriptions are correct, Grim clearly has not avoided the problem of presupposing universal quantification by appealing to non-denoting terms. And, of course, it is not obvious that Grim meta-philosophical caveats can be consistently formulated without quantification of the sort that he ends up repudiating.

To sum up, the metaphysical Cantorian core of the argument against a set of all truths is

flawed.
Cantor's theorem is asserted as either a mathematical or a metaphysical truth. If it is asserted to be a mathematical truth, then the assertion is at the very least ambiguous or incomplete since Cantor's theorem is valid in some mathematical theories and not in others. The attempted disproof of God's omniscience is therefore meta-mathematically inadequate insofar as it fails to take into account well-known mathematical contexts in which Cantor's theorem does not hold. More seriously, the disproof fails to acknowledge standard meta-mathematical conceptions which can, by analogy, be used to establish the relative consistency of certain theistic positions. If, on the other hand, 'Cantor's theorem' is supposed to refer to a metaphysical truth, then the atheistic argument begs the question. The metaphysical assertions about a set of all truths in the atheistic argument actually imply that Cantor's theorem is inapplicable to a set of all truths. Grim position is, moreover, ultimately philosophically untenable since the conclusions he want to draw from the "Cantorian" arguments--such as that "there is no set of all truths", "any proposition which entails that there is a set of all truths is false", "nothing instantiates an incoherent concept," and "there is no God"--cannot be asserted from the philosophical position in which he is forced to take refuge.

In the final section we will show that the above Cantorian argument is also historically uninformed since resources for answering the atheistic appropriation of Cantor's arguments are contained in Cantor's own writings about his philosophy of the infinite and its relation to theology.

Section 5. CANTOR'S THEOLOGY OF ABSOLUTE INFINITY.

In a letter to Dedekind dated August 31, 1899, Cantor himself proposed a Cantorian argument to prove the inconsistency of the notion of a 'system S of all thinkable classes'. Cantor, of course, did not see such arguments as obstacles to theistic belief. On the contrary, Cantor saw them as evidence of the truth of his mathematical theory of the Transfinite precisely because they comported well with his theology of Absolute infinity.

Cantor's philosophy of the infinite superseded that of his philosophical predecessors in at least two respects. In order to dispense with certain paradoxes involving completed infinities, Aristotle had distinguished between the potential and the actual infinite. Adopting Aristotle's distinction and assuming the theological premises that only God is actually and absolutely infinite and that it is impossible to study God's essence mathematically, Aquinas concluded that mathematicians could only study the potentially infinite [Summa Theologica, Part I, Question 7)]. Cantor's mathematical theory of the infinite, however, allowed him to transcend these philosophical limitations.

Within the category of the actual infinite Cantor introduced a distinction between (a) the increasable actual infinite or transfinite and (b) the unincreasable or Absolute actual infinite. Of the former category, Cantor wrote:

> In particular, there are transfinite cardinal numbers and transfinite ordinal types which, just as much as the finite numbers and forms, possess a definite mathematical uniformity, discoverable by men. All these modes of the transfinite have existed from eternity as ideas in the Divine intellect.

In a remarkable passage from a letter written as early as 1883, Cantor wrote the following about the latter category of the Absolute infinite:

> The Absolute can only be recognized, and never known, not even approximately.... The absolutely infinite sequence of numbers therefore seems to me in a certain sense a suitable symbol of the Absolute.

Given his distinction, Cantor could argue, contrary to Aquinas, that mathematicians could properly study actual transfinite infinities and still agree with Aquinas that the study of the Absolute infinite was beyond mere human comprehension.

Given Cantor's intuitive distinction between actual transfinite infinities and Absolute infinity, it is understandable why he was not more disturbed by the paradoxes. The Burali-Forti paradox concerning the existence of a set of all ordinals was first published by Burali-Forti in 1897, but Cantor had in fact already anticipated the problem by 1895. The ordinals are obtained by extending the sequence of natural numbers in such a way that the stages of the set-theoretical hierarchy are identified with sets. According to Cantor's conception, the class of all ordinals is clearly too large to be assigned any definitive cardinal number within the set-theoretical hierarchy and so the class of ordinals forms an absolutely infinite multiplicity.

Cantor proposed to avoid the Burali-Forti paradox by treating the collection of all ordinal numbers as an "inconsistent multiplicity", which he explained in a letter to Dedekind as follows:

> A collection [Vielheit] can be so constituted that the assumption of a 'unification' of all its elements into a whole leads to a contradiction, so that it is impossible to conceive of the collection as a unity, as a 'completed object'. Such collections I call absolute infinite or inconsistent collections.

Cantor's distinction between consistent and inconsistent multiplicities gave him, in effect, a set/proper class distinction which enabled him to deal with the set-theoretical paradoxes. From this perspective, we can perhaps see why Cantor remarked that the absolutely infinite class of all ordinals is "an appropriate symbol of the Absolute".

Cantorian views about the nature of Absolute infinity find intriguing mathematical support in the postulation of large cardinal axioms by contemporary set theorists. The axiom of inaccessible cardinals was first suggested by Ernst Zermelo in 1930, formulated by Alfred Tarski in 1938, and defended by Kurt Gödel in 1947. An uncountable cardinal I is strongly inaccessible if and only if (i) for all cardinals a <

l, l is not the sum of a cardinals less than l, or (ii) if a is any set with cardinality less than l, then the cardinality of P(a) is also less than l. Postulating the axiom of inaccessible cardinals to ZF is quite analogous to adding the axiom of infinity to ZF. In fact, if the restriction that õ be uncountable is omitted from the above definition, then the axiom can be seen as asserting, as does the axiom of infinity, the existence of the first (countable) inaccessible cardinal. The addition of this axiom to ZF has never been shown to lead to contradiction. It follows from Gödel’s theorem, in fact, that the addition of this axiom leads to a strictly stronger theory.

One justification for adopting large cardinal axioms fits particularly well with the philosophical and heuristic doctrines underlying Cantorian set theory. The reflection principle states that the universe V of sets is so complex that any attempt to structurally characterize the universe of sets will fail, and one will instead have characterized only a set of smaller rank. In other words, any attempt to uniquely describe the universe V also applies to some smaller Ra that “reflects” the property ascribed to V. Applying this Cantorian theme to ZF, we see that the class of ordinals in ZF is determined by the operations of addition and exponentiation. But given the reflection principle it cannot be that such an intuitively simple characterization yields the class of all ordinals (and so all the stages of the universe of set) but instead characterizes only a set.

The reflection principle synthesizes Cantor’s doctrine of Absolute infinity with Gödel’s suggestion that the plausibility of such axioms of infinity “shows clearly, not only that the axiomatic system of set theory as used today is incomplete, but also that it can be supplemented without arbitrariness by new axioms which only unfold the content of the concept of set”. The reflection principle can therefore be traced back to Cantor’s theory that the sequence of all transfinite numbers, like God, is absolutely infinite.

Stated in terms of inconceivability, Cantor’s theological notion of God is also characterized by a reflection principle. The reflection principle transposed into a principle of theology becomes: any attempt to conceive God will turn out to be equally applicable to some lesser being. Or perhaps more traditionally, “God is greater than anything we can conceive”.

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ENDNOTES

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Ibid., p. 98.

Ibid., p. 147.

Ibid., p. 91.

Cantor's famous result that no countable sequence of elements from the real interval can exhaust that interval was communicated in a letter to Dedekind in 1873. It was not until 1891, however, that Cantor discovered the simpler diagonal core of the proof for this result, which is now commonly referred to as 'Cantor's theorem'.

Grim, pp. 91-93. The argument, however, can be stated even more succinctly assuming the Schröder-Bernstein theorem:

> Assume there is a set T of all truths. For every distinct subset of T, there will be a distinct truth, namely the truth that that subset is a subset of truths. But Cantor's theorem implies there can be no 1-1 function from P(S) into S. Hence, there can be no set T of all truths.

Willard van Orman Quine, "New Foundations for Mathematical Logic", American Mathematical Monthly 44 (1937), pp. 70-80 and "On Cantor's Theorem", Journal of Symbolic Logic 2, no. 3, (Sept. 1937), pp. 120-124.

Was it irrational for Aristotle to believe in the existence of motion even though he did not have the conceptual resources to decisively refute Zeno's paradoxes? Would it be irrational to believe in one's free will if one could not answer the philosophical challenges posed by the perennial problem of free will and determinism? Would it be irrational for a physicist to trust quantum mechanical predictions even if she were unable to resolve the metaphysical paradoxes of quantum mechanics? It could be perfectly rational for a theist to believe in divine omniscience even in the face of unresolved philosophical challenges to its coherence.

It should be clear, in any case, that recasting the premises of the atheistic Cantorian argument in terms of a challenge to the coherence of theism would neither disprove God's existence nor the rationality of theism.

Alvin Plantinga, The Nature of Necessity, (New York: Oxford University Press, 1974), chapter 9.

It could be objected that there just isn't any such collection of basic truths with which to begin the iterative construction. However, for the purposes of a logical defense, we may take our initial set of truths to be any collection of truths the objector wants.

For philosophical discussions of the iterative notion of set, see George Boolos, "The Iterative Conception of Set", Hao Wang, "The Iterative Conception of Set", and Charles Parsons, "What is the Iterative Conception of Set?" reprinted in Benacerraf and Putnam (eds.), Philosophy of Mathematics (second edition), (New York: Cambridge University Press, 1983).

Christopher Menzel, "Theism, Platonism, and the Metaphysics of Mathematics", Faith and Philosophy 4, no. 4 (1987), pp. 365-382.

Here we do not address the further question of whether omniscience is adequately characterized in

terms of knowing a set of all true propositions. There could be propositions that are logically possible to know but which God cannot know since it is not logically possible that God know these propositions. If, for example, the de se proposition “I know what it is to sin”, as uttered by Adam is distinct from the proposition that “Adam knows what it is to sin”, then, assuming a plausible theory of indexicals, it would seem that God can know the latter, but not the former. If so, it would, of course, be pointless to define the notion of omniscience in terms of knowing all true propositions. Similar problems arise for other types of propositions that contain indexicals.

Neither do we address the question why propositional and not personal knowledge should be regarded as the paradigm of knowing.

When this more radical line of arguing against the coherence of quantification for ZF is taken, however, then the essential disagreement is not divine omniscience but a failure either to appreciate or to accept the standard resolution of Cantor’s paradox within ZF.

That the quantifiers range over the universe of all sets is not a truth within ZF, but is instead a meta-linguistic truth about the range of the quantifiers in the intended model of ZF. Some philosophers have argued that quantification in ZF does require quantification over a set. Charles Parsons in his “Sets and Classes”, Noûs 8 (1974), pp. 1-12, for example, has argued along modalized lines that the domain of quantifiers in ZF, while perhaps constituting an actual proper class is nonetheless a possible set.

Contemporary atheistic philosophers of religion have a tendency to ignore the rich context of theological and religious tradition and to place too much credence in their own stipulative definitions of divine attributes. At least one traditional theological doctrine, the doctrine of divine simplicity, for example, suggests that we shouldn’t expect to be able to characterize omniscience as knowledge of a set of truths anymore than we should expect to so characterize God. I am indebted to Arthur Howe for suggesting that the doctrine of divine simplicity could be used to defeat Cantorian arguments against omniscience and universal quantification.

Here we are not claiming that NF is a more intuitive and natural set theory than ZF. Instead, we consider NF as one possible way of modelling a universe of truths for which Cantor’s theorem fails.

Tyler Burge’s theory of truth in “Semantical Paradox” (Journal of Philosophy 76, (1979), pp. 169-198) can be seen as appropriating the NF technique of stratification as a basis for constructing a theory of truth in which the semantic paradoxes about truth are pragmatically resolved.

More precisely, for all x, x Î {y:j(y)} if and only if j(x), where the conditionj(y) is stratified and contains the variable y free and j(x) comes from j(y) by proper substitution of the variable x for y.

The technical reason why the argument for Cantor’s theorem fails in NF is that the condition for the diagonal set D = {x Î S: x Î f(x)} cannot be stratified. A function f(x) is typically treated in set theory as a set of ordered pairs. It turns out that a set of ordered pairs is stratified only if there is some assignment of numbers that assigns both members of the pair the same number. Hence if the condition

for D were stratified we must have that

the number of ‘x’ = the number of ‘f(x)’ .

On the other hand, the number of the term on the left hand side of ‘Î’ must be one greater than the number of the term of the right. Hence if the condition for D were stratified we must also have that

the number of ‘x’ + 1 = the number of ‘f(x)’ .

Since both of these conditions cannot hold simultaneously, the diagonal set D required for Cantor’s theorem does not exist.

This assumption of Grim can be disputed. See, for example, George Boolos “Nominalistic Platonism”, Philosophical Review 94, (1985), pp. 327-344.

Grim, p. 115.

Ibid., p. 123.

Michael Hallett, Cantorian Set Theory and the Limitation of Size (New York: Oxford University Press, 1984), p. 167.

Cantor’s 1895 Letter to Jeiler quoted by Hallett, p. 21.

Quoted in Hallett, p. 42.

When Cantor discovered that Saint Augustine had also endorsed the view that the infinity of numbers exists as eternal ideas in God but are incomprehensible to us due to our permanent and ineradicable imperfect understanding, he went so far as to footnote an entire excerpt from Augustine’s De civitate Dei, Chapter 19, Book XII entitled “Contra eos, qui dicunt ea, quae infinita sunt, nec Dei posse scientia comprehendi” (“Against those who say that not even God’s knowledge can comprehend things that are infinitely many”). See Dauben, p. 229; the translation used here is by Norman Kretzman.

Quoted in Dauben, p. 245.

See Christopher Menzel’s “Cantor and the Burali-Forti Paradox”, The Monist 67 (1984), pp. 92-105. Cantor, in fact, attempted to demonstrate that collections such as the collection of cardinal numbers were also inconsistent multiplicities by finding a subset of the collection equinumerous to the set of all ordinals. In adopting this methodological principle, Hallett (p. 45) argues that Cantor transformed his notion of absolute inconsistent infinities into a theory of limitation of size.

Ernst Zermelo, “Über Grenzzahlen und Mengenbereiche: neue Untersuchungen über die Grundlagen der Mengenlehre”, Fundamenta Mathematicae 16 (1930), pp. 29-47. Alfred Tarski, “Über unerreichbare Kardinalzahlen”, Fundamenta Mathematicae 30 (1938), pp. 68-69. Kurt Gödel, “What is Cantor’s Continuum Problem?” American Mathematical Monthly 54 (1947), pp. 515-512 (reprinted in Benacerraf and Putnam (eds.), pp. 258-273).

For a history of the emergence of the axiom of inaccessible cardinals see K. Kanamori and M. Magidor’s “The Evolution of Large Cardinal Axioms in Set Theory”, Higher Set Theory: Proceedings, Oberwolfach, Germany (G. H. Muller and D. S. Scott, eds.), Lecture Notes in Mathematics 669 (New York: Springer-Verlag, 1978), pp. 99-275 and also see Penelope Maddy’s “Believing the Axioms.

I",
Journal of Symbolic Logic 53, no. 2 (1988), pp. 481-511.
18 If an inaccessible cardinal I exists, then (i) it is possible to obtain a model of ZF among the sets of
cardinality less than I. However, it is one of the consequences of Gödel's theorem that (ii) one cannot
prove the consistency of a foundational system by methods expressible in that system. Therefore since
the existence of a model for ZF implies ZF is consistent, (i) and (ii) imply that the existence of inaccessible cardinals cannot be proved within ZF.
Gödel, in Benacerraf and Putnam (eds.), pp. 476-477.
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